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VIBRATING MEMBRANE PROBLEM SOLVED USING SPLINE COLLOCATION METHOD WITH DIRICHLET CONDITIONS

Nilesh A. Patel[1] & Jigisha U. Pandya[2]

[1]Department of Mathematics, Shankersinh Vaghela Bapu Institute of Technology, Gandhinagar, Gujarat, India

[2]Department of Mathematics, Sarvajanik college of Engineering & Technology, Surat, Gujarat, India

ABSTRACT

In this paper, two-dimensional equation, subject to Dirichlet initial boundary conditions is presented, and a Spline Collocation Method is utilized for solving the problem. Also, Spline provides continuous solution, in contrast to another method, which only provides discrete approximations. It is found that this method is a powerful mathematical tool and can be applied to a large class of linear and nonlinear problems in different fields of science and technology. Numerical results obtained by the present method are in good agreement with the analytical solutions available in the literature.

KEYWORDS: Spline Collocation Method, Partial Differential Equations, Two Dimensional Hyperbolic Equation & Dirichlet Boundary Conditions

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Original Article

INTRODUCTION

Recently, new numerical methods have gained the interest of researchers for finding numerical solutions to partial differential equations. This interest was driven by the need for applications, both in industry and sciences. Theory and numerical methods for solving initial boundary value problems were investigated by many researchers. In the last decade, there has been a growing interest in the new numerical techniques for linear and non-linear initial boundary value problems. In the field of partial differential equations, boundary value problem is a differential equation, together with a set of additional restraints, called the boundary conditions. A solution to a boundary value problem is a solution to the differential equation, which also satisfies the boundary conditions. Boundary value problems arise in engineering, applied mathematics and several branches.

However, it is usually difficult to obtain closed-form solutions for boundary value problems, especially for non-linear boundary value problems. In most cases, only approximate solutions (either numerical solution or analytical solutions) can be expected. Some numerical methods such as finite difference method, finite element method, spline approximation method, Jacobi's Method and SOR method have been developed for obtaining numerical solutions to boundary value problems.

In this paper, we apply a spline collocation method approach, to obtain a solution to the wider class of non-linear systems of boundary value problems. The spline method approach is widely utilized for the numerical solution of non-linear problems arising in real world applications.

The governing equations here are highly partial differential equations, which are solved by using the spline collocation method. In this way, the paper has been organized as follows. In section 2, we use the spline

collocation method, in section 3 spline solution of explicit method and implicit method. The conclusions are summarized in section 4.

SPLINE COLLOCATION METHOD

The general form of hyperbolic PDE with one space variables x

and time variable t is given by

$$u_{tt} = c^2 (u_{xx} + u_{yy}) \tag{i}$$

with a Dirichilet boundary conditions, namely

$$u(0, y, t) = u(x, 0, t) = \tag{ii}$$

and two initial conditions at t = 0 (Cauchy conditions)

$$u(x, y, 0) = f(x, y)$$

$$u_t (x, y, 0) = g(x, y). \tag{iii}$$

In equation (i), c^2 is a constant term; it depends upon some physical quantities in case of different types of problems.

Divide the region $0 \leq x \leq a$ into say n sun-intervals each of width Δx such that $n\Delta x = a$, $0 \leq y \leq b$ into say m sun-intervals each of width Δy such that $n\Delta y = b$.

The subscript k denotes the time and (I, j) denote the position. For simplicity, consider square region, i.e. a = b = L the length of the region,

m = n and $\Delta x = \Delta y = h$.

The points of subdivisions are (x_i, yi), i,j = 0(1)n.

For explicit scheme, the formula is obtained in a similar manner as discussed in section (3.9), by replacing the right side by twice, the second derivative of cubic spline S (x) at the (i, j) th mesh point I. e. $2S''(x_{i,j})$.

Let ui,j we get

$$\mathbf{(u_{i-1,j,k+1} + 4u_{i,j,k+1} + u_{i+1,j,k+1})}$$

$$\mathbf{= (2+12r^2)\, u_{i+1,j,k} + (8-24r^2)\, u_{i,j,k} + (2+12r^2)\, u_{i-1,j,k}}$$

$$\mathbf{-[\, u_{i+1,j,k-1} + 4u_{i,j,k-1} + u_{i-1,j,k-1}]} \tag{iv}$$

Where $\mathbf{r = c\Delta t/h}$ i,j = 1(1)n-1

The values of $u_{i,j,-1}$, i,j = 1(1) n -1 are obtained from following relation

$$u_{i,j,-1} = u_{i,j,-1} - 2g(x_i,y_j)\Delta t \tag{v}$$

The equation (iv) is known as a cubic spline explicit formula to solve hyperbolic type PDE of the form given by

the equation (i).

The values obtained from equation (v) using initial and boundary conditions, the equation (iv) gives a system of (n-1) x (n-1) simultaneous equations in (n-1) x (n-1) unknowns.

After calculating the results for k = 0; the results for k = 1, 2,…. are obtained in a similar manner.

$r \leq (1/2)$ is the required condition for convergence and stability of this explicit method.

Likewise, implicit scheme for obtaining the solution of parabolic PDE with one space variable, we have an implicit scheme to solve hyperbolic differential equation with two space variables.

This implicit scheme is unconditionally stable.

The cubic spline implicit formula to solve equation (i) is as follows, by replacing the right side of equation (i)

$$(1-6r^2)u_{i+1,j,k+1} + (4+12r^2)u_{i,j,k+1} + (1-6r^2)u_{i-1,j,k+1}$$

$$= (6r^2-1)\, u_{i+1,j,k-1} + (4+12r^2)\, u_{i,j,k-1} + (6r^2-1)\, u_{i-1,j,k-1}$$

$$+2[\, u_{i+1,j,k} + 4u_{i,j,k} + u_{i-1,j,k}] \qquad \text{(vi)}$$

Where $r = c\Delta t/h \qquad i,j = 1(1)n-1$

Due to similar discussions as above, also equation (vi) gives (n-1) x (n-1) simultaneous equations in (n-1) x (n-1) are unknowns.

After calculating results for $(k+1)^{th}$ time level, the results for $(k+2)^{th}$ time level are obtained in similar manner by repeating the process.

The problem of vibrating membrane is well-known to researchers. Assume that membrane is tightly stretched and homogeneous i.e. its mass per unit area is constant, it is perfectly flexible and is so thin that, it offers no resistance for bending.

The membrane is stretched and then fixed along its entire boundary in the xy – plane, and the tension T caused by stretching the membrane is the same at every point in all directions and does not change during the motion.

The deflections u (x, y, t) of the membrane during the motion are small compared to the size of the membranes.

Consider the forces acting on a small portion of the membrane, as shown in below figure.

Since the deflection of the membrane and the angles of inclination are small, the sides of the portion may be taken approximately equal to Δx and Δy.

The tension T is the force per unit length. Forces acting on the edges are $T\Delta x$ and $T\Delta y$, which could be taken tangent to the membrane, since the membrane is perfectly elastic.

Let the force $T\Delta y$ make angles A and B with the horizontal on the opposite edges of the membrane, since the membrane is perfectly elastic.

The resultant vertical component of force due to $T\Delta y$ is therefore,

$$(T\Delta y)\sin B-(T\Delta y)\sin A$$

$(T\Delta y)\tan B-(T\Delta y)\tan A$

$= T\Delta y\{(\partial u/\partial x)_{x+\Delta x}-(\partial u/\partial x)_x\}$

$= T\Delta y\{(\partial^2 u/\partial x^2\}\Delta x$

Up to a first order approximation, note that sine have been replaced by tangent, because, the angles A and B are small.

Similarly, the forces $T\Delta x$ acting on the edges of length Δx can be shown to have the vertical component.

$T\Delta x\{(\partial^2 u/\partial y^2\}\Delta y$

If m is the mass per unit area of the membrane, by Newton's second law of motion

$(m\Delta x\Delta y)\ \partial^2 u/\partial t^2 = T\ \{(\partial^2 u/\partial x^2+\partial^2 u/\partial y^2\}\Delta x\Delta y$

$\partial^2 u/\partial t^2 = c^2\{(\partial^2 u/\partial x^2+\partial^2 u/\partial y^2\}$ where $c^2 = T/m$.(vii)

The above equation (vii) is hyperbolic PDE in two space variables x and y, and time variable t.

The solution for the case of a rectangular membrane is given below by cubic spline explicit, as well as an implicit method, using the following initial and boundary conditions.

Dirichilet boundary conditions:

$u(0,y,t) = 0\ 0 \le y \le t$

$u(x,0,t) = 0\ 0 \le x \le t$ (viii)

Initial conditions at t = 0 (Cauchy condition)

$u(x,y,0) = f(x,y) = \sin\pi x.\sin\pi y\ ;\ 0 \le x,y \le 1$ (ix)

$u(x,y,0) = 0$

SPLINE SOLUTIONS WITH EXPLICIT METHOD

Using initial and boundary conditions are described by equations (viii) and (ix), the solution of equation (vii) is obtained through explicit formula given by the equation (iv) as follows.

Let the region $0 \le x \le 1$ and $0 \le y \le 1$ be divided into 10 sub-intervals, each of length h = 0.1. Let c = 0.1 and Δt = 0.01, these gives r = 0.01. After calculating the values of $u_{i,\ j-1}$, i, j = 1(1) n form equation (v), using initial condition, substitute the values of $u_{i,\ j-1}$, $2 + 12r^2$ and $8 - 24\ r^2$ into the equation (iv) one gets,

For k = 0

For j = 1

i = 1 $u_{0,\ 1,\ 1} + 4u_{1,\ 1,\ 1} + u_{2,\ 1,\ 1} = 0.563599$

Since $u_{0,\ 1,\ 1} = 0$

$4u_{1,\ 1,\ 1} + u_{2,\ 1,\ 1} = 0.563599$

i = 2 $u_{1,\ 1,\ 1} + 4u_{2,\ 1,\ 1} + u_{3,\ 1,\ 1} = 1.072029$

$i = 3\ u_{2,1,1} + 4u_{3,1,1} + u_{4,1,1} = 1.475521$

$i = 4\ u_{3,1,1} + 4u_{4,1,1} + u_{5,1,1} = 1.734579$

$i = 5\ u_{4,1,1} + 4u_{5,1,1} + u_{6,1,1} = 1.873844$

Since $u_{4,1,1} = u_{6,1,1}$ (Due to symmetry of the problem)

$2u_{4,1,1} + 4u_{5,1,1} = 1.873844$

In a similar manner, calculate the equation j = 1 (1)9 one gets 9 x 9 simultaneous linear equations in 9 x 9 unknowns.

This system of equations can be solved by any well-known method. Once the values of u are known at the first level of time, the process can be repeated for second time level and so on. The results obtained by explicit method are given as follows. Due to the symmetry of the problem, results are given for $0 \le y \le 0.5$, $0 \le x \le 0.5$ in the table (a) and (b).

The results obtained at y = 0.1 are plotted in the figures (P).

Table 1: Velocity Distribution in Rectangular Vibrating Membrane through Cubic Spline Explicit Method u→ at t=0.01

x\y	0	0.1	0.2	0.3	0.4	0.5
0	0	0	0	0	0	0
0.1	0	0.095491	0.181634	0.249998	0.293890	0.309014
0.2	0	0.181634	0.345488	0.475523	0.559012	0.587779
0.3	0	0.249998	0.475523	0.654502	0.769413	0.809009
0.4	0	0.293890	0.559012	0.769413	0.904499	0.951047
0.5	0	0.309014	0.587779	0.809009	0.951047	0.999990

Table 2: Velocity Distribution in Rectangular Vibrating Membrane through Cubic Spline Explicit method u → at t=0.03

x\y	0	0.1	0.2	0.3	0.4	0.5
0	0	0	0	0	0	0
0.1	0	0.095483	0.181619	0.249978	0.293867	0.308989
0.2	0	0.181619	0.345461	0.475485	0.558967	0.587732
0.3	0	0.249978	0.475485	0.654450	0.769352	0.808944
0.4	0	0.293867	0.558967	0.769352	0.904427	0.950917
0.5	0	0.308989	0.587732	0.808944	0.950971	0.999910

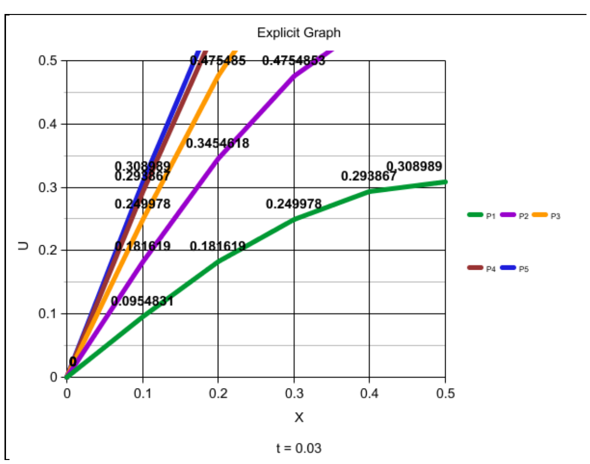

Figure 1

SPLINE SOLUTIONS WITH IMPLICIT METHOD

In similar manner discussed as earlier, using initial and boundary condition at $\Delta t = 0.01$, $r = 0.01$, the solution of equation (i) is obtained by using cubic spline implicit formula given by equation (vi) as follows.

For k = 0

For j = 1

i = 1 $(4.0012)\, u_{1,1,1} + (0.9994)\, u_{2,1,1} = 0.563602$

I = 2 $(0.9994)\, u_{1,1,1} + (4.0012)\, u_{2,1,1} + (0.9994)\, u_{3,1,1} = 1.072034$

i = 3 $(0.9994)\, u_{2,1,1} + (4.0012)\, u_{3,1,1} + (0.9994)\, u_{4,1,1} = 1.475528$

I = 4 $(0.9994)\, u_{3,1,1} + (4.0012)\, u_{4,1,1} + (0.9994)\, u_{5,1,1} = 1.734587$

i = 5 $(1.9988)\, u_{4,1,1} + (4.0012)\, u_{5,1,1} = 1.823853$

One gets system of 9x9 simultaneous linear equations in 9x9 unknowns.

Once the values of u are obtained at the first time level, the values for second time level are obtained in similar manner by repeating the process.

Results obtained by cubic spline implicit method are given at different times in the below tables.

The results are plotted in the below figures at y = 0.1.

Table 3: Velocity Distribution in Rectangular Vibrating Membrane through Cubic Spline Implicit Method u → at t = 0.01

x\y	0	0.1	0.2	0.3	0.4	0.5
0	0	0	0	0	0	0
0.1	0	0.095491	0.181634	0.249998	0.293890	0.309014
0.2	0	0.181634	0.345488	0.475523	0.559012	0.587779
0.3	0	0.249998	0.475523	0.654502	0.769413	0.809009
0.4	0	0.293890	0.559012	0.769413	0.904499	0.951047
0.5	0	0.309014	0.587779	0.809009	0.951047	0.999990

Table 4: Velocity Distribution in Rectangular Vibrating Membrane through Cubic Spline Implicit Method u →at t = 0.03

x\y	0	0.1	0.2	0.3	0.4	0.5
0	0	0	0	0	0	0
0.1	0	0.095491	0.181634	0.249998	0.293890	0.309014
0.2	0	0.181634	0.345488	0.475523	0.559012	0.587779
0.3	0	0.249998	0.475523	0.654502	0.769413	0.809009
0.4	0	0.293890	0.559012	0.769413	0.904499	0.951047
0.5	0	0.309014	0.587779	0.809009	0.951047	0.999990

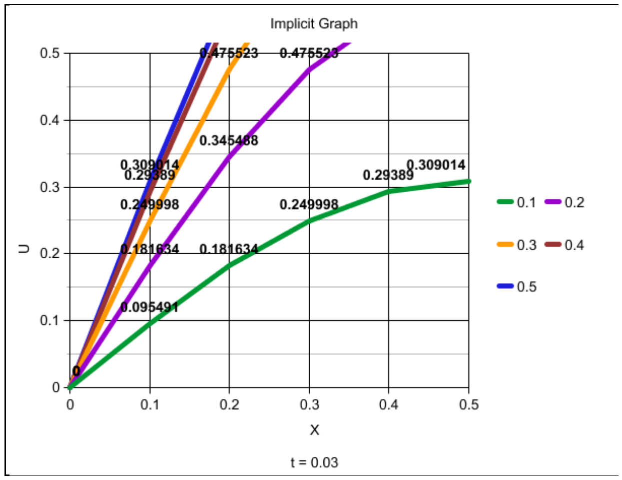

Figure 2

DISCUSSIONS OF RESULTS

The velocity distribution obtained from the solution of equation (i) by cubic spline explicit as well as implicit methods are compared with exact solutions at y = 0.1 for t = 0.03 in the above table.

From the table, it is clear that the solutions obtained by both these methods are accurate, up to five digits of decimal places.

The figure indicates analysis, which compares the exact solutions with spline solutions obtained by both the methods, explicit as well as implicit at t = 0.03, clearly revealing that the spline solutions are quite accurate and reliable.

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